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1 author:

Andrea Stupňanová
Slovak University of Technology in Bratislava

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Special fuzzy measures on infinite countable sets and related aggregation functions

Andrea Stupňanová

Department of Mathematics, Slovak University of Technology, Radlinského 11, 81368 Bratislava, Slovakia

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Abstract

While both additive and symmetric fuzzy measures on a finite universe are completely described by a probability distribution vector, this is no more the case of a countably infinite universe. After a brief discussion of additive fuzzy measures on positive integers, we characterize all symmetric fuzzy measures on integers by means of three constants and of two probability distribution vectors. OWA operators for n arguments were introduced by Yager in 1988. Grabisch in 1995 has shown representation of OWA operators by means of Choquet integral with respect to symmetric normed capacities. Based on symmetric capacities on positive integers, we extend the concept of OWA operators to infinitary sequences and thus we develop the concept of infinitary OWA operators.

Keywords: Additive fuzzy measure; Symmetric fuzzy measure; k-Order additivity; p-Symmetry; OWA operator; Choquet integral

1. Introduction

OWA operators belong to frequently applied aggregation functions in several engineering and sociological domains. They were introduced by Yager [21] in 1988 to aggregate finitely many inputs into one representative value, generalizing the minimum, maximum and arithmetic mean functions. The symmetry of OWA operators makes them invariant under permutation of input arguments. However, this property causes problems when trying to introduce OWA operators for infinite sequences (see [13] for a deep discussion of infinitary aggregation functions). As observed in [7], each OWA operator can be represented as a Choquet integral with respect to a symmetric fuzzy measure (capacity). This observation was an inspiration for our introduction of infinitary OWA operators in this contribution, including the discussion of special fuzzy measures on infinite countable spaces $\mathcal{X}$.

Recall that for $n \in \mathbb{N}$, an n-ary function $A : [0, 1]^n \to [0, 1]$ is called an aggregation function [9] whenever it is monotone in each coordinate and $A(0, \dots, 0) = 0$, $A(1, \dots, 1) = 1$. Similarly, a function $A : [0, 1]^{\mathbb{N}} \to [0, 1]$ is called an aggregation function whenever it is monotone in each coordinate and $A(0, \dots, 0, \dots) = 0$, $A(1, \dots, 1, \dots) = 1$. For more details we recommend a recent monograph [9]. An aggregation function $A : [0, 1]^n \to [0, 1]$ ($A : [0, 1]^{\mathbb{N}} \to [0, 1]$) is symmetric whenever $A(x_1, \dots, x_n) = A(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any $(x_1, \dots, x_n) \in [0, 1]^n$ and σ a permutation of $(1, \dots, n)$, i.e., $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is a bijection ($A((x_n)_{n\in\mathbb{N}}) = A((x_{\sigma(n)})_{n\in\mathbb{N}})$ for any $(x_n)_{n\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}$ and any bijection $\sigma : \mathbb{N} \to \mathbb{N}$).

E-mail address: andrea.stupnanova@stuba.sk.

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In our paper we will work also with the property of comonotone additivity. Note that $A : [0,1]^n \to [0,1]$ is comonotone additive whenever $A(\bar{x}+\bar{y}) = A(\bar{x}) + A(\bar{y})$ for any $\bar{x}, \bar{y} \in [0,1]^n$ such that $\bar{x}+\bar{y} \in [0,1]^n$, and $\bar{x}$ and $\bar{y}$ are comonotone, i.e., $(x_i - x_j)(y_i - y_j) \geq 0$ for any $i, j \in \{1, \ldots, n\}$. This property can be straightforwardly extended also to the case of infinitary aggregation function $A : [0,1]^{\mathbb{N}} \to [0,1]$. Due to Schmeidler [18], comonotone additive aggregation functions are just Choquet integrals with respect to fuzzy measures. Note that for $\mathcal{X} = \{1, \ldots, n\}$ or $\mathcal{X} = \mathbb{N}$, a set function $m : 2^{\mathcal{X}} \to [0,1]$ is called a fuzzy measure (capacity, monotone measure) whenever $m(\emptyset) = 0$, $m(\mathcal{X}) = 1$ and $A \subset B \subseteq \mathcal{X}$ implies $m(A) \leq m(B)$.

Each infinite countable set $\mathcal{X}$ is isomorphic to the set $\mathbb{N} = \{1, 2, \ldots\}$ of all positive integers, and thus since now we will consider $\mathbb{N}$ as our universe.

2. Special fuzzy measures on countable infinite sets

A fuzzy measure is a monotone set function satisfying the boundary conditions.

Definition 1. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space. A mapping $m : \mathcal{A} \to [0,1]$ which is nondecreasing, and $m(\emptyset) = 0$, $m(\mathcal{X}) = 1$, is called a fuzzy measure.

Two special types of fuzzy measures are often applied in several applications, especially in multicriteria decision making.

Definition 2. A fuzzy measure m defined on a measurable space $(\mathcal{X}, \mathcal{A})$ is called

- Additive whenever $m(A \cup B) = m(A) + m(B)$ for all $A, B \in \mathcal{A}$, $A \cap B = \emptyset$.
- Symmetric if $m(A) = m(B)$ whenever there is a bijection $\varphi : \mathcal{X} \to \mathcal{X}$ such that $B = \varphi(A) = \{\varphi(x) | x \in A\}$.

In the case of a finite universe $\mathcal{X}$ (with cardinality $|\mathcal{X}| = n$) we always deal with $\mathcal{A} = 2^{\mathcal{X}}$. Then the symmetry of a fuzzy measure m on $\mathcal{X}$ means $m(A) = m(B)$ whenever $|A| = |B|$. Evidently, to describe a symmetric fuzzy measure m in such a case means that we should know the values of $m(A)$ for cardinality $|A| = 1, 2, \ldots, n-1$ ($m(A) = 0$ if $|A| = 0$ and $m(A) = 1$ if $|A| = n$ by Definition 1).

Due to the monotonicity of m, we have to determine values $u_i = m(A)$, $|A| = i$, $i = 1, \ldots, n-1$ such that $0 = u_0 \leq u_1 \leq \cdots \leq u_{n-1} \leq u_n = 1$. Put $w_i = u_i - u_{i-1}, i = 1, \ldots, n$. Then $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, i.e., $(w_1, \ldots, w_n)$ is a probability distribution vector on $\mathcal{X}$. Recall that then $m(A) = \sum_{i=1}^{|A|} w_i$, with convention that an empty sum equals 0.

Similarly, if $\mathcal{X} = \{x_1, \ldots, x_n\}$ and if m is additive, it is enough to put $w_i = m(\{x_i\})$. Then again $(w_1, \ldots, w_n)$ is a probability distribution vector and $m(A) = \sum_{x_i \in A} w_i$. The system $\mathcal{W} = \{(w_1, \ldots, w_n) | w_i \geq 0, \sum_{i=1}^{n} w_i = 1\}$ of probability distribution vectors is a convex class with base $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$.

Note that consequently $\mathcal{D}_n = \{m | m$ is additive fuzzy measure on $\mathcal{X} = \{1, \ldots, n\}\}$ and $\mathcal{S}_n = \{m | m$ is symmetric fuzzy measure on $\mathcal{X} = \{1, \ldots, n\}\}$ are convex classes with base $\delta_{\{1\}}, \ldots, \delta_{\{n\}}$ (Dirac measures) and with base $s_1, \ldots, s_n$, where

$$s_i(A) = \begin{cases} 1 & \text{if } |A| \geq i, \\ 0 & \text{else,} \end{cases} \quad \text{respectively.}$$

Observe that $s_1 = m^*$ is the strongest fuzzy measure while $s_n = m_*$ is the weakest fuzzy measure on $\mathcal{X} = \{1, \ldots, n\}$. Recall that the only fuzzy measure on this $\mathcal{X}$ which is both additive and symmetric is related to the uniform distribution and it is given by $m(A) = |A|/n$.

Recently, Mesiar et al. [14,11,12], see also [20], have proposed and discussed the concept of universal fuzzy measures M as a system $M = (m_n)_{n \in \mathbb{N}}$ of fuzzy measures m_n on $\mathcal{X}_n = \{1, \ldots, n\}$ with some additional properties. One way how to define a universal fuzzy measure was based on a fuzzy measure m defined on $(\mathbb{N}, 2^{\mathbb{N}})$, putting $m_n(A) = m(A)/m(\mathcal{X}_n)$ whenever $A \subseteq \mathcal{X}_n$.

However, the fuzzy measures on $(\mathbb{N}, 2^{\mathbb{N}})$ were not discussed in mentioned works [14,11,12,20], neither in other papers exploiting the notion of universal fuzzy measures, such as [16,3].

2.1. Additive fuzzy measures on $\mathbb{N}$

For the description of additive fuzzy measures on $\mathbb{N}$, their continuity is crucial. While in the case of finite $\mathcal{X}$, each additive fuzzy measure m on $\mathcal{X}$ is, in fact, a probability measure, in the case $\mathcal{X} = \mathbb{N}$ this is true only if we require the continuity of discussed additive fuzzy measure m.

Definition 3. Let $(A_n)_{n\in\mathbb{N}}$ be a system of measurable sets from $\mathcal{A}$. A fuzzy measure m on $(\mathcal{X}, \mathcal{A})$ is

- Continuous if for any system $(A_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} A_n = A$ it holds $m(A) = \lim_{n\to\infty} m(A_n)$.
- Continuous from below if $(A_n)_{n\in\mathbb{N}}$ is a nondecreasing system and $m(\bigcup_n A_n) = \sup_n m(A_n)$.
- Continuous from above if $(A_n)_{n\in\mathbb{N}}$ is a nonincreasing system and $m(\bigcap_n A_n) = \inf_n m(A_n)$.

Note that if m is additive, then the three types of continuity introduced in Definition 3 are equivalent. However, then for $\mathcal{X} = \mathbb{N}$ and $A \subseteq \mathbb{N}$ it is enough to put $A_n = A \cap \{1, \ldots, n\}$. Putting $w_i = m(\{i\}),\ i \in \mathbb{N}$, the additivity of m ensures $m(A_n) = \sum_{i\in A_n} w_i$, while the continuity from below of m ensures

$$m(A) = \sup_n m(A_n) = \sum_{i\in A} w_i. \tag{1}$$

Especially $1 = m(\mathbb{N}) = \sum_{i=1}^{\infty} w_i$.

Hence, each continuous additive fuzzy measure m on $\mathbb{N}$ is described via (1) by means of a probability distribution vector $(w_1, w_2, \ldots)$, and thus $m = \sum_{i=1}^{\infty} w_i \delta_{\{i\}}$ is a convex sum of Dirac measures.

However, if the continuity of an additive fuzzy measure m on $\mathbb{N}$ is not guaranteed, m can be rather peculiar. Observe that due to Tarski [19], see also [1], there exists an additive $\{0, 1\}$-valued fuzzy measure m on $\mathbb{N}$ such that $m(A) = 0$ for any finite set $A \subset \mathbb{N}$. Note that then $m(A) = 1$ for any A with finite complement A^C, while if both A and A^C are infinite, then $m(A) \in \{0, 1\}$ (and then $m(A^C) = 1 - m(A)$). A general description of additive fuzzy measures on $\mathbb{N}$ is still an open problem.

2.2. Symmetric fuzzy measures on $\mathbb{N}$

Definition 4. Two subsets $A, B \subseteq \mathcal{X}$ are called $\mathcal{X}$-isomorphic if there is a bijection $\varphi : \mathcal{X} \to \mathcal{X}$ such that $\varphi(A) = B$.

Recall that symmetric fuzzy measures on $\mathcal{X}$ are just those fuzzy measures which do not distinguish $\mathcal{X}$-isomorphic sets. Hence the symmetric fuzzy measures on $\mathcal{X}$ are related to the classes of equivalence $\overset{\mathcal{X}}{\sim}$ on $2^{\mathcal{X}}$ given by $A \overset{\mathcal{X}}{\sim} B$ if and only if A is $\mathcal{X}$-isomorphic to B.

Proposition 1. *The universe $\mathbb{N}$ has the following classes of $\overset{\mathbb{N}}{\sim}$ equivalence*:

(i) $\mathcal{A}_n = \{A \subseteq \mathbb{N} | |A| = n\}$ *with* $n = 0, 1, 2, \ldots$.
(ii) $\mathcal{B}_n = \{A \subseteq \mathbb{N} | |A^C| = n\}$ *with* $n = 0, 1, 2, \ldots$.
(iii) $\mathcal{C} = \{A \subseteq \mathbb{N} | A \text{ and } A^C \ are\ infinite\}$.

Proof. The only nontrivial case is (iii).

Suppose that A, B are infinite subsets of $\mathbb{N}$ such that also A^C and B^C are infinite. Then

$$A = \{a_n\}_{n\in\mathbb{N}}, \quad B = \{b_n\}_{n\in\mathbb{N}}, \quad A^C = \{c_n\}_{n\in\mathbb{N}}, \quad B^C = \{d_n\}_{n\in\mathbb{N}},$$

where $A \cup A^C = B \cup B^C = \mathbb{N}$, $A \cap A^C = B \cap B^C = \emptyset$, and all sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}, \{d_n\}_{n\in\mathbb{N}}$ are strictly increasing.

To see that $A \overset{\mathbb{N}}{\sim} B$ it is enough to define

$$\varphi : \mathbb{N} \to \mathbb{N} \text{ by } \varphi(a_n) = b_n \quad \text{and} \quad \varphi(c_n) = d_n, \ \ n \in \mathbb{N}.$$

Proposition 1 is a basic tool to describe symmetric fuzzy measures on $\mathbb{N}$, taking into account their monotonicity. □

Theorem 1. *A set function $m : 2^{\mathbb{N}} \to [0, 1]$ is a symmetric fuzzy measure on $\mathbb{N}$ if and only if there is a nondecreasing sequence $(\alpha_n)_{n=0}^{\infty}$ with $\alpha_0 = 0$ and $\sup_n \alpha_n = \alpha \le 1$, a nonincreasing sequence $(\beta_n)_{n=0}^{\infty}$ with $\beta_0 = 1$ and $\inf_n \beta_n = \beta \ge \alpha$ and a constant $\gamma \in [\alpha, \beta]$ so that*

$$m(A) = \begin{cases} \alpha_n & \text{if } |A| = n, \\ \beta_n & \text{if } |A^C| = n, \\ \gamma & \text{else.} \end{cases} \tag{2}$$

The proof follows from Proposition 1.

The strongest fuzzy measure m^* on $\mathbb{N}$ is a symmetric fuzzy measure characterized by $\alpha_1 = 1$. Similarly, the weakest fuzzy measure m_* on $\mathbb{N}$ is characterized by $\beta_1 = 0$. Two distinguished symmetric fuzzy measures on $\mathbb{N}$ are $m^{(*)}$ and $m_{(*)}$ characterized, respectively, by $\alpha = 0, \gamma = 1$ and by $\beta = 1, \gamma = 0$, i.e.,

$$m^{(*)}(A) = \begin{cases} 0 & \text{if } A \text{ is finite,} \\ 1 & \text{else,} \end{cases} \qquad m_{(*)}(A) = \begin{cases} 1 & \text{if } A^C \text{ is finite,} \\ 0 & \text{else.} \end{cases}$$

Theorem 2. *A symmetric fuzzy measure m on $\mathbb{N}$ is continuous from below (continuous from above) if and only if $\alpha = 1$ ($\beta = 0$). Consequently, there is no continuous symmetric fuzzy measure on $\mathbb{N}$.*

Proof. Let m be a symmetric fuzzy measure on $\mathbb{N}$. If $\alpha = 1$ and $A_n \nearrow A$, then either A is finite and hence there is $n_0 \in \mathbb{N}$ such that $A_m = A$ for all $m \ge n_0$, i.e., $\lim_{n\to\infty} m(A_n) = m(A_{n_0}) = m(A)$, or A is infinite. In the later case, $m(A) = 1$ and $\lim_{n\to\infty} |A_n| = |A| = \infty$ and hence $\lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} \alpha_{|A_n|} = \alpha = 1$, proving the continuity from below of m.

Vice versa, if m is continuous from below, put $A_n = \{1, \ldots, n\}$. Then $A_n \nearrow \mathbb{N}$, i.e., $\alpha = \sup_n \alpha_n = \lim_{n\to\infty} m(A_n) = m(\mathbb{N}) = 1$.

The case of the continuity from above is similar. □

Theorem 2 indicates that any symmetric fuzzy measure m on $\mathbb{N}$ which is continuous from below (from above) is characterized by a probability distribution vector $(w_1, w_2, \ldots)$ so that

$$m(A) = \sum_{i=1}^{|A|} w_i, \quad \text{i.e.,} \quad \alpha_n = \sum_{i=1}^{n} w_i; \; n \in \mathbb{N},$$

$$\left(m(A) = 1 - \sum_{i=1}^{|A^C|} w_i, \; \text{i.e.,} \; \beta_n = 1 - \sum_{i=1}^{n} w_i; \; n \in \mathbb{N} \right).$$

Formula (2) and Theorem 2 allow another look on symmetric fuzzy measures on $\mathbb{N}$.

Corollary 1. *A set function $m : 2^{\mathbb{N}} \to [0, 1]$ is a symmetric fuzzy measure on $\mathbb{N}$ if and only if m is a convex sum of a symmetric fuzzy measure m_1 continuous from below, of a symmetric fuzzy measure m_2 continuous from above, and of $m^{(*)}$ and $m_{(*)}$,*

$$m = \alpha m_1 + (\gamma - \alpha) m^{(*)} + (\beta - \gamma) m_{(*)} + (1 - \beta) m_2. \tag{3}$$

Hence each symmetric fuzzy measure m on $\mathbb{N}$ is characterized by two probability distribution vectors $(p_1, p_2, \ldots)$ and $(q_1, q_2, \ldots)$, and three constants $0 \le \alpha \le \gamma \le \beta \le 1$, so that, comparing (3) with (2), it holds $\alpha_n = \alpha \cdot p_n$, and $\beta_n = 1 - (1 - \beta) q_n$, $n = 0, 1, 2, \ldots$.

2.3. *k-Order additivity and p-symmetry of fuzzy measures on $\mathbb{N}$*

Let $\mathcal{X}$ be a finite non-empty set. We recall the Möbius transform M_m of a fuzzy measure m defined by

$$M_m(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} m(B), \quad \text{(G.-C. Rotta)}, \tag{4}$$

where $|A \setminus B|$ is the cardinality of relative complement of B in A. Note also that a set function m can be reconstructed from M_m by

$$m(A) = \sum_{B \subset A} M_m(B) \quad \textit{the Zeta transform}. \tag{5}$$

Definition 5 (*Grabisch [8]*). Let $k \in \mathbb{N}$. A fuzzy measure m defined on a finite measurable space $(\mathcal{X}, \mathcal{A})$ is called k-order additive fuzzy measure whenever $M_m(A) = 0$ for all $A \subset \mathcal{X}$ with cardinality $|A| > k$.

This definition cannot be directly extended for countable space $\mathcal{X} = \mathbb{N}$. However, based on ideas from [10] k-additivity of a fuzzy measure m on $\mathbb{N}$ means that there is an additive set function M on $\mathbb{N}^k$ such that $m(A) = M(A^k)$ for all $A \subseteq \mathbb{N}$. If we require the continuity of fuzzy measure m, for $k = 1$ (m is a continuous additive fuzzy measure), we get the same representation as in Section 2.1:

$$m = \sum_{i=1}^{\infty} w_i \delta_{\{i\}} \quad \text{where } w_i = m(\{i\}) = M_m(\{i\}).$$

Similarly, each 2-additive continuous fuzzy measure m can be written as a linear combination (see Mesiar [10])

$$m = \sum_{i=1}^{\infty} w_i \delta_{\{i\}} + \sum_{\{i,j\}, i \neq j} w_{\{i,j\}} \delta_{\{i,j\}}, \tag{6}$$

where the generalized Dirac measure $\delta_{\{i,j\}}$ for $i \neq j$ is given by

$$\delta_{\{i,j\}}(A) = \begin{cases} 1 & \text{if } \{i, j\} \subset A, \\ 0 & \text{else}, \end{cases}$$

and $w_{\{i,j\}} = M_m(\{i, j\}, \{j, i\})$ are such that for any $A \subsetneq \mathbb{N}$ and $l \in \mathbb{N} \setminus A$ the inequality $\sum_{i \in A} w_{\{i,l\}} + w_l \geq 0$ holds that means that $m(A) \leq m(A \cup \{l\})$. Note that $w_{\{i,j\}} \in]-\infty, \infty[$.

As well as k-additive fuzzy measures generalize the additive fuzzy measures, there are p-symmetric fuzzy measures generalizing symmetric fuzzy measures [15]. We modify the notion of a p-symmetric fuzzy measure on $\mathbb{N}$ based on special permutations of $\mathbb{N}$.

Definition 6. Let $p \in \mathbb{N}$ be fixed. A fuzzy measure $m : 2^{\mathbb{N}} \to [0, 1]$ is p-symmetric whenever there is a partition $(V_1, \ldots, V_p)$ on $\mathbb{N}$ so that $m(A) = m(\varphi(A))$ for any permutation $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\varphi(V_i) = V_i$, $i = 1, \ldots, p$.

Note that the original definition [15] of p-symmetry applicable to finite universes $\mathcal{X}$ only is based on the cardinality of sets $A \cap V_1, \ldots, A \cap V_p$. Our approach allows to extend the notion of p-symmetry to any (measurable) universe.

It is evident, that $m(V_i) = a_i \geq 0$ for $i = 1, \ldots, p$. There exists an aggregation function $B : [0, 1]^p \to [0, 1]$ such that $B(a_1, \ldots, a_p) = 1$ and $m(A) = B(m(A \cap V_1), \ldots, m(A \cap V_p))$. If V_i is finite, i.e., $|V_i| = n_i$, then there exist numbers a_{ij}, $j \in \{0, \ldots, n_i\}$, $a_{ij} \leq a_{ij+1}$, $a_{i0} = 0$, $a_{in_i} = a_i$ such that $m(A \cap V_i) = a_{i|A \cap V_i|}$. For infinite V_i (such case always exist) if $a_i > 0$, we get a symmetric fuzzy measure m_i on V_i, $m_i(A) = m(A \cap V_i)/a_i$.

Summarizing,

$$m(A) = B(a_1 m_1(A \cap V_1), \ldots, a_p m_p(A \cap V_p)),$$

where m_i is a symmetric fuzzy measure on V_i for all $i = 1, \ldots, p$. We introduce an aggregation operator $C : [0, 1]^p \to [0, 1]$, $C(x_1, \ldots, x_p) = B(a_1 x_1, \ldots, a_p x_p)$.

We have the following representation of a p-symmetric fuzzy measure m:

Theorem 3. *A set function $m : 2^{\mathbb{N}} \to [0, 1]$ is a p-symmetric fuzzy measure on $\mathbb{N}$ if and only if there exist m_i—the symmetric fuzzy measures on V_i, respectively, for all $i = 1, \ldots, p$, and m can be expressed by*

$$m(A) = C(m_1(A \cap V_1), \ldots, m_p(A \cap V_p)), \tag{7}$$

where $C : [0, 1]^p \to [0, 1]$ is an aggregation operator.

Example 1 (*2-symmetric*). Let $C(x, y) = (x + 2y)/3$, $V_1 = \{2n - 1 | n \in \mathbb{N}\}$, $V_2 = \{2n | n \in \mathbb{N}\}$ and $m_1 = m^{(*)}$, $m_2 = m_{(*)}$.

Then

$$m(A) = \begin{cases} 0 & \text{if } A \cap V_1 \text{ is finite \& } A^C \cap V_2 \text{ is infinite,} \\ \dfrac{1}{3} & \text{if } A \cap V_1 \text{ is infinite \& } A^C \cap V_2 \text{ is infinite,} \\ \dfrac{2}{3} & \text{if } A \cap V_1 \text{ is finite \& } A^C \cap V_2 \text{ is finite,} \\ 1 & \text{if } A \cap V_1 \text{ is infinite \& } A^C \cap V_2 \text{ is finite.} \end{cases}$$

3. Integral based infinitary aggregation functions

Recall that Choquet integral based on a fuzzy measure m on $(\mathcal{X}, \mathcal{A})$ from a measurable function $f : \mathcal{X} \to [0, 1]$ is given by [2,4,5,17]

$$Ch(m, f) = \int_0^1 m(\{f \geq t\})\, dt, \tag{8}$$

where the right-hand side of (8) is the Riemann integral. OWA operators were introduced by Yager [21] as aggregation functions $\mathrm{OWA}_{\mathbf{w}} : [0, 1]^n \to [0, 1]$ given by

$$\mathrm{OWA}_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{\sigma(i)}, \tag{9}$$

where $\mathbf{w} \in [0, 1]^n$ is an a priori given weight vector, $\sum_{i=1}^n w_i = 1$, and σ is a permutation such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$. Axiomatic characterization of OWA operators is simple—they are exactly symmetric comonotone additive aggregation functions. Therefore, an OWA operator is just Choquet integral based on a symmetric fuzzy measure m on $\mathcal{X} = \{1, \ldots, n\}$ such that $m(A) = \sum_{i=1}^{\mathrm{card(A)}} w_{n-i+1}$, see [7]. Vice versa, Choquet integral on $\mathcal{X} = \{1, \ldots, n\}$ based on a symmetric fuzzy measure m is just an OWA operator with a weight vector $\mathbf{w}$, $w_i = m(\{i, \ldots, n\}) - m(\{i+1, \ldots, n\})$.

For infinite sequences $\bar{x} \in [0, 1]^{\mathbb{N}}$ a straightforward extension of formula (4) into

$$\mathrm{OWA}_{\mathbf{w}}(\bar{x}) = \sum_{i=1}^{\infty} w_i x_{\sigma(i)} \tag{10}$$

is not well defined once there is any accumulation point of $\bar{x}$ different from $\sup x_i$, as then the permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $(x_{\sigma(i)})_{i \in \mathbb{N}}$ is a nondecreasing sequence does not exist. However, axiomatic approach and its representation by Choquet integral are valid also in infinitary case.

Definition 7. A comonotone additive symmetric aggregation function $\mathrm{OWA} : [0, 1]^{\mathbb{N}} \to [0, 1]$ is called an (infinitary) OWA operator.

Due to Schmeidler [18], see also [2], we have the next representation of infinitary OWA operators.

Theorem 4. *A mapping* $\mathrm{OWA} : [0, 1]^{\mathbb{N}} \to [0, 1]$ *is an infinitary OWA operator if and only if there is a symmetric fuzzy measure m on* $\mathbb{N}$ *such that* $\mathrm{OWA}(\bar{x}) = Ch(m, \bar{x})$.

As a consequence of Theorem 4 and Corollary 1 we have the next result generalizing formula (9).

Corollary 2. *A mapping* $\mathrm{OWA} : [0, 1]^{\mathbb{N}} \to [0, 1]$ *is an infinitary OWA operator if and only if there are constants* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1]$, $\sum_{i=1}^4 \lambda_i = 1$ *and infinitary weight vectors* $\mathbf{p}, \mathbf{q} \in [0, 1]^{\mathbb{N}}$, $\sum_{i \in \mathbb{N}} p_i = \sum_{i \in \mathbb{N}} q_i = 1$, *so that*

$$\mathrm{OWA}(\bar{x}) = \lambda_1 \sum_{i=1}^{\infty} p_i x_{(i)} + \lambda_2 \liminf \bar{x} + \lambda_3 \limsup \bar{x} + \lambda_4 \sum_{i=1}^{\infty} q_i x^{(i)}, \tag{11}$$

where $x_{(i)}$ ($x^{(i)}$) *is the i-th smallest (greatest) element of* $\bar{x} = (x_i)_{i\in\mathbb{N}}$ *if it exists, and* $x_{(i)} = \liminf \bar{x} = \lim_{i\to\infty} \inf\{x_i, x_{i+1}, \ldots\}$ ($x^{(i)} = \limsup \bar{x} = \lim_{i\to\infty} \sup\{x_i, x_{i+1}, \ldots\}$) *if the i-th smallest (greatest) element of* $\bar{x}$ *does not exist.*

Observe that due to Theorem 2, an infinitary OWA operator is continuous from below (from above) if and only if $\lambda_1 = 1$ ($\lambda_4 = 1$), and that there is no continuous infinitary OWA operator.

Remark 1.

(i) Formula (11) can be extended for inputs from $\mathbb{R} =]-\infty, \infty[$, supposing the boundedness of $\bar{x}$ in $\mathbb{R}$. In such case the asymmetric Choquet integral [5,17] should be applied in Choquet integral representation, compare Theorem 3.

(ii) There is no symmetric additive fuzzy measure m on $\mathbb{N}$, and thus there is no version of the arithmetic mean on $[0,1]^{\mathbb{N}}$ possessing the same properties as the classical arithmetic mean on $[0,1]^{\mathbb{N}}$. See [6].

Example 2.

(i) Extremal OWA operators are related to extremal fuzzy measures. Thus the weakest infinitary OWA operator $\mathrm{OWA}_* = \inf$ (i.e., $\lambda_1 = 1$, $p_1 = 1$), while the strongest OWA operator $\mathrm{OWA}^* = \sup$ (i.e., $\lambda_4 = 1$, $q_1 = 1$).

(ii) For an infinitary OWA characterized by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, $\mathbf{p}$ and $\mathbf{q}$, if $\bar{x} = (x_i)_{i\in\mathbb{N}}$ is an increasing sequence then

$$\mathrm{OWA}(\bar{x}) = \lambda_1 \sum_{i=1}^{\infty} p_i x_i + (1-\lambda_1)\sup \bar{x}.$$

Similarly, if $\bar{x}$ is a decreasing sequence then

$$\mathrm{OWA}(\bar{x}) = \lambda_4 \sum_{i=1}^{\infty} q_i x_i + (1-\lambda_4)\inf \bar{x}.$$

(iii) For an infinitary OWA characterized by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, $\mathbf{p}$ and $\mathbf{q}$, its dual OWA^d given by $\mathrm{OWA}^d(\bar{x}) = 1-\mathrm{OWA}(\mathbf{1}-\bar{x})$ is characterized by $\lambda_4, \lambda_3, \lambda_2, \lambda_1$, $\mathbf{q}$ and $\mathbf{p}$. Consequently, an infinitary OWA is selfdual, i.e.,

$$\mathrm{OWA}(\bar{x}) = 1-\mathrm{OWA}(\mathbf{1}-\bar{x}) \quad \text{if and only if } \mathbf{p}=\mathbf{q},\ \lambda_1=\lambda_4 \text{ and } \lambda_2=\lambda_3.$$

(iv) For $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and $\mathbf{p}, \mathbf{q}$ with $p_1 = q_1 = 1$,

$$\mathrm{OWA}(\bar{x}) = \tfrac{1}{4}(\inf \bar{x} + \liminf \bar{x} + \limsup \bar{x} + \sup \bar{x}).$$

4. Concluding remarks

After a discussion of particular fuzzy measures on infinite countable universes (here represented by $\mathbb{N}$), we have introduced and discussed infinitary OWA operators aggregating infinite sequences. Our approach is based on properties characterization (symmetry and comonotone additivity), compare [13], and it cannot be obtained as a limit of n-ary OWA operators. Recall that even for the arithmetic mean, limit of partial arithmetic means (Cesaro means) need not exist, and even if it exist it is not symmetric [3,6].

Note that duality of fuzzy measures, $m^d(A) = 1 - m(A^C)$, leads to a genuine property of additive fuzzy measures—selfduality, $m = m^d$.

In the case of symmetric fuzzy measures on $\mathbb{N}$, based on (3), it is not difficult to check that a symmetric fuzzy measure m on $\mathbb{N}$ is selfdual if and only if

$$m = \alpha m_1 + (\tfrac{1}{2}-\alpha)m^{(*)} + (\tfrac{1}{2}-\alpha)m_{(*)} + \alpha m_1^d = \alpha(m_1 + m_1^d) + (\tfrac{1}{2}-\alpha)(m^{(*)} + m_{(*)}),$$

where $\alpha \in \left[0, \tfrac{1}{2}\right]$ and m_1 is a symmetric fuzzy measure on $\mathbb{N}$ continuous from below.

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