A RIESZ TYPE REPRESENTATION THEOREM FOR RIESZ SPACE-VALUED POSITIVE LINEAR MAPPINGS (Nonlinear Analysis and Convex Analysis)

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A RIESZ TYPE REPRESENTATION THEOREM FOR RIESZ SPACE-VALUED POSITIVE LINEAR MAPPINGS

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ABSTRACT. Let X be a completely regular Hausdorff space and V a Dedekind complete Riesz space. The purpose of this note is to give a necessary and sufficient condition (tightness condition) which assures the validity of an analogue of the Riesz representation theorem for a positive linear mapping from $C(X)$ into V.

1. INTRODUCTION

Let X be a Hausdorff space and V a Dedekind complete Riesz space. Denote by $\mathcal{B}(X)$ the σ-field of all Borel subsets of X. A V-valued σ-measure on X is a finitely additive set function $\mu : \mathcal{B}(X) \to V$ such that $\mu(\bigcup_{n=1}^{\infty} A_n) = \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \mu(A_k)$ whenever $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{B}(X)$. If V possesses a Hausdorff vector topology τ for which each upper bounded monotone increasing sequence in V converges in the τ-topology to its least upper bound, V-valued σ-measures are ordinary topological vector space-valued measures that are fairly well understood; see Diestel and Uhl [2] and Kluvánek and Knowles [4]. But V need not possess any such topology; see Floyd [3].

The purpose of this note is to give a necessary and sufficient condition which assures that a given positive linear mapping T from $C(X)$, the space of all bounded, continuous, real-valued functions on X, into a Dedekind complete Riesz space V can be uniquely represented by a V-valued σ-measure μ on X such that $T(f) = \int_X f d\mu$ for all $f \in C(X)$. A successful analogue of the Riesz representation theorem was first proved by Wright [8, Theorem 4.1] and [10, Theorem 4.5] in the case that X is compact. See also [9, Theorem 1] for the case that X is locally compact. For the case that the representing measure μ is finitely additive, see Lipecki [5] and the literature therein. In Boccuto and Sambucini [1] a version of the above representation theorems has been discussed for "monotone integrals" with respect to Dedekind complete Riesz space-valued capacities.

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In Section 2 we recall some basic facts on Riesz spaces and give some preliminary results concerning regularities of Riesz space-valued σ-measures on a topological space. The results explained in the preceding paragraph are obtained in Section 3.

2. Notation and preliminaries

All the topological spaces in this paper are Hausdorff and denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all natural numbers respectively.

2.1. **Riesz spaces.** A Riesz space is said to be *Dedekind complete* if every non-empty order bounded subset has a least upper bound. Every Dedekind complete Riesz space is Archimedean; see Schaefer [6, page 54].

Let V be a Riesz space and put $V^+ := \{u \in V : u \geq 0\}$. Given a net $\{u_\alpha\}_{\alpha\in\Gamma}$ in V we write $u_\alpha \downarrow u$ to mean that it is a decreasing net and $\inf_{\alpha\in\Gamma} u_\alpha = u$. The meaning of $u_\alpha \uparrow u$ is analogous.

Let $e \in V$ with $e > 0$. Denote by V_e the principal ideal generated by e, that is, $V_e := \{u \in V : |u| \leq re \text{ for some } r > 0\}$. Then, V_e is an AM-space with order unit e under the order unit norm $\|u\|_e := \inf\{r > 0 : |u| \leq re\}$, so that by the Kakutani-Krein theorem (see, for instance, [6, page 104]), it is isometrically and lattice isomorphic to $C(S)$, the space of all (bounded) continuous real-valued functions on a compact space S. Since V is Dedekind complete, so also is V_e. Hence S is Stonean, that is, the closure of every open subset of S is also open [6, page 108].

2.2. **σ-measures.** Let X be a topological space. Denote by $\mathcal{B}(X)$ the σ-field of all Borel subsets of X, that is, the σ-field generated by the open subsets of X. Denote by $C(X)$ the Banach lattice of all bounded, continuous, real-valued functions on X with supremum norm $\|f\|_\infty := \sup_{x\in X} |f(x)|$ and by $B(X)$ the Banach lattice of all Borel measurable, bounded, real-valued functions on X with the same norm.

Let V be a Dedekind complete Riesz space. A finitely additive, positive set function $\mu : \mathcal{B}(X) \to V$ is called a *σ-measure* on X if it is σ-additive in the sense that whenever $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{B}(X)$ then $\mu(\bigcup_{n=1}^\infty A_n) = \sup_{n\in\mathbb{N}} \sum_{k=1}^n \mu(A_k)$. We emphasize that only measures taking positive values are considered in this paper.

As in the scalar case, every σ-measure has the monotone sequential continuity from above and from below, that is, whenever $\{A_n\}_{n\in\mathbb{N}}$ is an increasing (respectively a decreasing) sequence of sets in $\mathcal{B}(X)$ then $\mu(\bigcup_{n=1}^\infty A_n) = \sup_{n\in\mathbb{N}} \mu(A_n)$ (respectively $\mu(\bigcap_{n=1}^\infty A_n) = \inf_{n\in\mathbb{N}} \mu(A_n)$).

In Wright [8, 10] a V-valued integral with respect to a σ-measure μ is constructed and the successful analogues of the monotone convergence theorem and the Lebesgue convergence theorem are obtained. We shall use the results there freely in this paper.

2.3. **Regularities of σ-measures.** As in usual measure theory on topological spaces we need to introduce some notions of regularities for Riesz space-valued σ-measures. Let X be a topological space and V a Dedekind complete Riesz space.

Definition 1. Let μ be a V-valued σ-measure on X.

(i) μ is said to be *quasi-regular* if whenever G is an open subset of X then

$$\mu(G) = \sup\{\mu(F) : F \subset G \text{ and } F \text{ is closed}\}.$$

(ii) μ is said to be *quasi-Radon* if whenever G is an open subset of X then

$$\mu(G) = \sup\{\mu(K) : K \subset G \text{ and } K \text{ is compact}\},$$

and it is said to be *tight* if the above condition holds for $G = X$.

(iii) μ is said to be *τ-smooth* if whenever $\{G_\alpha\}_{\alpha\in\Gamma}$ is an increasing net of open subsets of X with $G = \bigcup_{\alpha\in\Gamma} G_\alpha$ then $\mu(G) = \sup_{\alpha\in\Gamma}\mu(G_\alpha)$.

Lemma 1. *Let μ be a V-valued σ-measure on X.*

(i) *μ is quasi-regular if and only if for each open subset G of X there exist a net $\{p_\alpha\}_{\alpha\in\Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{F_\alpha\}_{\alpha\in\Gamma}$ of closed subsets of X such that $F_\alpha \subset G$ and $\mu(G - F_\alpha) \le p_\alpha$ for all $\alpha \in \Gamma$.*

(ii) *μ is quasi-Radon if and only if for each open subset G of X there exist a net $\{p_\alpha\}_{\alpha\in\Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of X such that $K_\alpha \subset G$ and $\mu(G - K_\alpha) \le p_\alpha$ for all $\alpha \in \Gamma$.*

(iii) *μ is tight if and only if there exist a net $\{p_\alpha\}_{\alpha\in\Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of X such that $\mu(X - K_\alpha) \le p_\alpha$ for all $\alpha \in \Gamma$.*

Further, the above nets $\{F_\alpha\}_{\alpha\in\Gamma}$ and $\{K_\alpha\}_{\alpha\in\Gamma}$ can be chosen to be increasing.

Lemma 2. *Let μ be a V-valued σ-measure on X. Then the following two conditions are equivalent:*

(i) *μ is tight and quasi-regular.*

(ii) *μ is quasi-Radon.*

Lemma 3. *Every quasi-Radon V-valued σ-measure μ on X is τ-smooth.*

The following result can be proved as in the case of scalar measures; see for instance [7, Proposition I.3.2].

Proposition 1. *Let μ be a τ-smooth V-valued σ-measure on X. Let $\{f_\alpha\}_{\alpha\in\Gamma}$ be a uniformly bounded, increasing net of lower semicontinuous real-valued functions on X. If $f = \sup_{\alpha\in\Gamma} f_\alpha$ is the pointwise supremum of f_α, then $\int_X f d\mu = \sup_{\alpha\in\Gamma}\int_X f_\alpha d\mu$.*

Lemma 4. *Assume that X is completely regular. Let μ and ν be τ-smooth V-valued σ-measures on X. If $\int_X f d\mu = \int_X f d\nu$ for each $f \in C(X)$ then $\mu = \nu$ on $\mathcal{B}(X)$.*

3. An analogue of the Riesz representation theorem

Let X be a topological space and V a Dedekind complete Riesz space. In this section we give a necessary and sufficient condition (tightness condition) which assures the validity of an analogue of the Riesz representation theorem for a positive linear mapping from $C(X)$ into V.

First we extend Proposition 4.1 [8] to the case that X is not necessarily compact.

Proposition 2. *Let X be a completely regular space and Y a compact space. Let $T : C(X) \to C(Y)$ be a positive linear mapping. Assume that there exist a net $\{p_\alpha\}_{\alpha\in\Gamma}$ in $C(Y)$ with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of X such that $T(f) \leq p_\alpha$ whenever $\alpha \in \Gamma$ and $f \in C(X)$ with $0 \leq f \leq 1$ and $f(K_\alpha) = \{0\}$. Put $N := \{y \in Y : \inf_{\alpha\in\Gamma} p_\alpha(y) > 0\}$. Then there exists a mapping $\tilde{T} : B(X) \to B(Y)$ such that*

(i) *$\tilde{T}$ is positive and linear,*
(ii) *for each $f \in C(X)$, $\tilde{T}(f)(y) = T(f)(y)$ for all $y \notin N$,*
(iii) *if $\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $B(X)$ which converges pointwise to f, then $f \in B(X)$ and*

$$\tilde{T}(f)(y) = \lim_{n\to\infty} \tilde{T}(f_n)(y) \text{ for all } y \in Y,$$

(iv) *if f is a lower semicontinuous real-valued function on X, then*

$$\tilde{T}(f)(y) = \sup\{T(g)(y) : 0 \leq g \leq f, g \in C(X)\} \text{ for all } y \notin N,$$

and hence $\tilde{T}(f)$ is lower semicontinuous on $Y - N$.

From Proposition 2 we naturally reach the following definition.

Definition 2. Let X be a topological space and V a Riesz space. We say that a positive linear mapping $T : C(X) \to V$ satisfies the *tightness condition* if there exist a net $\{p_\alpha\}_{\alpha\in\Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha\in\Gamma}$ of compact subsets of X such that $T(f) \leq p_\alpha$ whenever $\alpha \in \Gamma$ and $f \in C(X)$ with $0 \leq f \leq 1$ and $f(K_\alpha) = \{0\}$.

Let S be a compact Stonean space. Denote by $\mathcal{M}$ the σ-ideal of all meager Borel subsets of S. Let κ be a canonical $C(S)$-valued σ-measure on S such that

(κ1) $\mathcal{M}$ is the kernel of κ,
(κ2) $\kappa(E) = \chi_E$ for all clopen subset E of S.

The existence of κ follows from [8, page 118] and κ is called the *Birkhoff-Ulam $C(S)$-valued σ-measure* on S.

The following lemma has been already given in [8] implicitly.

Lemma 5. *Let κ be the Birkhoff-Ulam $C(S)$-valued σ-measure on S. Then $\int_S f d\kappa = f$ for all $f \in C(S)$.*

We are now ready to give an analogue of the Riesz representation theorem for a Dedekind complete Riesz space-valued positive linear mapping.

Theorem 1. *Let X be a completely regular space and V a Dedekind complete Riesz space. Let $T : C(X) \to V$ be a positive linear mapping. Then the following two conditions are equivalent:*

(i) *T satisfies the tightness condition.*

(ii) *There exists a quasi-Radon V-valued σ-measure μ on X such that*

$$T(f) = \int_X f d\mu \quad \text{for all } f \in C(X). \tag{1}$$

Further, the μ is determined by (1) *and the quasi-Radonness of μ.*

The tightness condition in the above theorem is automatically satisfied if X is compact, and hence Theorem 1 reduces to a special case of the results obtained in [8, Theorem4.1] and [10, Theorem 4.5]. See also [9, Theorem 1]. However, our work will be needed to develop the theory of the weak order convergence of Riesz space-valued σ-measures, in which we usually assume that the involved σ-measures are defined on metric spaces or more generally on completely regular spaces. As an application in this light, we shall show in a later work that the operation making the Borel product of two Riesz space-valued σ-measures is jointly continuous with respect to the weak order convergence of σ-measures.

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